sugar solutions without taking up the sugar at the same time supports the author's theory that the property of selective permeability is (in the case of colloidal membranes) a result of preferential adsorption.

It is also shown in the above paper that the side of a membrane in contact with pure water has a greater moisture content than the side in contact with sugar solution. This fact supports the hypothesis—first advanced by Graham on experimental grounds—that osmosis across the membrane takes place because pure water induces a greater moisture pressure and concentration inside the membrane than the solution does.

On the Ordinary Convergence of Restricted Fourier Series.

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§ 1. The necessary and sufficient condition that a trigonometrical series* should be a Fourier series is that the integrated series should converge to an integral throughout the closed interval of periodicity, and should be the Fourier series, accordingly, of an integral. Conversely, starting with the Fourier series of an integral and differentiating it term by term, we obtain the Fourier series of the most general type, namely, one associated with any function possessing an absolutely convergent integral.

If the Fourier series which is differentiated is not the Fourier series of an integral, but of a function which fails to be an integral, at even a single point, the derived series will not be a Fourier series.

The series so obtained, namely, by differentiating term-by-term the Fourier series of a function which, without being an integral in the whole closed interval of periodicity, is an integral in one or more sub-intervals, I have found it convenient in a recent paper† to call "restricted Fourier series." I retain the term "Fourier," because such series possess, in the interval or

* It is convenient to suppose the trigonometrical series to have no constant term, or to regard the integrated series as a series obtained from the original series by interchanging the coefficients of $\sin nx$ and $\cos nx$, changing the sign of one of them, and dividing both by n. The integral to which the integrated series converges is accordingly a periodic function. By integral we mean, as usual, absolutely convergent or Lebesgue integral.

† "On the Convergence of the Derived Series of Fourier Series," read at the June Meeting, 1916, of the L.M.S. and recently presented to that Society.

intervals to which they are "restricted," several of the properties of Fourier series, so that they can be employed in analysis in some of the ways in which ordinary Fourier series have been used.

There is, however, one important difference between the new and the old types of Fourier series. In dealing with the former, we can in general only employ the method of summation due to Cesàro, index unity at least. The theorems which continue to hold for restricted Fourier series in the interval to which they are restricted are obtained by modifying suitably theorems relating to Fourier series in which Cesàro convergence occurs. For more information on this point I must refer my readers to the paper just cited. It will suffice for our present purposes to refer to only one of these properties. It is known that the behaviour of a Fourier series as regards convergence, or mode of oscillation, at a particular point x of the interval $(-\pi, \pi)$ of periodicity, depends only on the nature of the associated function in the neighbourhood of the point, this neighbourhood being taken as small as we please, and the method of summing the series, or of obtaining its upper and lower functions, peak and chasm functions, etc., being the ordinary classical one.

It is remarkable both that this theorem fails in the case of restricted Fourier series, and that it becomes true when the summation is performed in the Cesàro manner, index ≥ 1.

Thus tests for the ordinary convergence of a restricted Fourier series of a type analogous to the known ones for Fourier series cannot exist, but only tests for the Cesàro convergence analogous to those for the Cesáro convergence of ordinary Fourier series.

And again only those theorems relating to the integration term-by-term of a Fourier series, after multiplication by another function, in which the convergence employed is of the Cesàro type, have their analogues in the general theory of restricted Fourier series.

To secure the ordinary convergence of a restricted Fourier series at a point or in an interval, and the holding good of integration theorems which involve ordinary convergence in an interval, conditions must be fulfilled which concern not only the neighbourhood of the point, or the interval considered, but also the whole of the rest of the interval of periodicity. In other words, additional hypotheses of a novel type will have to be made. In the present communication to the Society I propose to initiate the new theory which thus arises, by considering a case which, though special in character, is probably the most important of all the cases that can present themselves. I assume that the failure of the derived series to be a Fourier series is due to the fact that, at a finite number of points, the function associated with the series from which it is derived fails to be an integral. In other words this function is an integral

over every sub-interval which does not contain any of the points in question, and over such intervals only.

The main general auxiliary theorem obtained is that *if* $\mathrm{F}(t)$ *is the function associated with the original Fourier series, and has accordingly* $f(t)$, *the function associated with the restricted Fourier series, for its differential coefficient almost everywhere, then the restricted Fourier series behaves in every interval not containing one of the points* $k_1, k_2, \ldots, k_s$ *at which* $\mathrm{F}(t)$ *ceases to be an integral, exactly like a Fourier series, provided :*

(1) $(x-k_r)\,\mathrm{F}(x)$ *is an integral in some interval containing* k_r, *for* $r = 1, 2, \ldots, s$*; and converges to zero, as* $x \to k_r$*;*

(2) *if* $q_r(x)$ *denote any function which, except in a certain sub-interval of the interval* $(-\pi, \pi)$, *surrounding* $x = k_r$, *is a periodic integral, and in that exceptional sub-interval is equal to* $\mathrm{F}(x)$, *then the coefficients of the derived series of the Fourier series of* $q_r(x)$ *converge to zero, for* $r = 1, 2, \ldots, s$.

It will be noticed that, in the case of a single exceptional point k_r, the second of the pair of conditions, thus shown to be sufficient, is certainly a necessary condition for a trigonometrical series to converge ; its terms must, like those of any other series, themselves converge to zero. Moreover, as is well known, if this convergence holds throughout an interval, however small, the coefficients must themselves converge to zero.

With regard to the first condition, I show in the present paper that all known conditions imposed upon $f(x)$, to secure the fulfilment of (2), involve at the same time the truth of (1). The conditions in question are those which I have recently communicated to the Society* ; they lead at once, among others, to the following theorems :—

If as $u \to 0$, *we have for* $r = 1, 2, \ldots, s$,

(i) $$\mathrm{F}(k_r+u)-\mathrm{F}(k_r-u) \to 0,$$

(ii) $$\mathrm{F}(u+k_r)-\mathrm{F}(u-k_r)-\frac{1}{u}\int_0^u \mathrm{F}(t+k_r)\,dt+\frac{1}{u}\int_0^u \mathrm{F}(t-k_r)\,dt \to 0,$$

and if further

(iii) $$\mathrm{F}(u+k_r)-\frac{1}{u}\int_0^u \mathrm{F}(t+k_r)\,dt$$

is an integral for some interval containing $u = 0$, $\mathrm{F}(x)$ *being the function associated with the Fourier series, from which the restricted Fourier series was derived, then the behaviour of the latter series at any point other than one of*

* "On the Order of Magnitude of the Coefficients of a Fourier Series," 'Roy. Soc. Proc.,' A, vol. 93, p. 42. For simplicity, the conditions are stated separately for an odd and an even function, and the singular point is supposed to be at the origin. The transition to the case contemplated in the present paper is of course immediate.

the exceptional points $k_1, k_2, \ldots, k_s$, *depends only on the nature of* $\mathrm{F}(x)$ *in the neighbourhood of the point considered, and, therefore, only on the function* $f(x)$ [$= \mathrm{F}'(x)$] *associated with the restricted Fourier series.*

Again, if, as $u \to 0$, we have for $r = 1, 2, \ldots, s$,

(i) $$\mathrm{F}(u+k_r)-\mathrm{F}(k_r-u) \to 0,$$

(ii) $$\mathrm{Lt}\, u\,[f(k_r+u)] = \mathrm{Lt}\, u\,[f(k_r-u)],$$

both these limits accordingly existing,

(iii) $$\frac{1}{u}\int_0^u \left| \frac{d}{dt}\left\{t^2 f(t+k_r)\right\} \right| dt$$

is a bounded function of u, *or, more generally,*

(iii*a*) $$\frac{1}{u}\int_0^u \left| d\left\{t^2 f(t+k_r)\right\} \right|$$

is a bounded function of u; *the notation being as in the preceding theorem, then the same conclusion follows.*

§ 2. These theorems take a very simple form if we specialise the character of the function further. Thus, we may, for example, replace the conditions (i), (ii), and (iii) of the former of these two theorems by the single condition that $uf(u)$ is an even function of bounded variation which has the unique limit zero, as $u \to 0$, and thus obtain immediately, among others, the theorem:—

If $uf(u)$ *is an even function of bounded variation in the whole interval* $(-\pi, \pi)$ *of periodicity, and has the unique limit zero, as* $u \to 0$, *then the restricted Fourier series associated with* $f(x)$ *converges ordinarily at all points of the interval.*

We may also replace the corresponding conditions in the second theorem by the condition that $u^2 f'(u)$ *is an even function, which approaches zero, as* $u \to 0$, *while* $f(u)$ *is itself a function of bounded variation in every interval not containing the origin*, and so obtain the theorem that *the restricted Fourier series associated with such a function* $f(x)$ *converges everywhere, except at the origin.*

Moreover, at the different points $k_1, k_2, \ldots, k_s$, of the fundamental theorem, we may, of course, secure the fulfilment of the corresponding condition by hypotheses, varying with the value of the subscript.

Closely connected with the subject of the paper are certain theorems depending on the uniform convergence to zero of that portion of the integral representing the nth partial summation, the treatment of which forms the main difficulty in these investigations. This uniformity is secured, like the convergence itself, by the hypotheses made. These additional theorems are those alluded to at the beginning of this introduction; their statement and proof will be found below.

§ 3. We begin by explaining some of the circumstances under which we may differentiate an integral of the form

$$\int_a^b f(x \pm u)\, g(u)\, du$$

under the sign of integration. In the succeeding theorems we encounter two such cases:—

(i) *When $f(t)$ has a differential coefficient at every point of the interval (a, b), and this differential coefficient is a bounded function of t, while $g(t)$ is summable.*

We may then multiply the bounded sequence

$$\{f(x+h \pm u) - f(x \pm u)\}/h$$

by $g(u)$ and integrate term by term, which gives us the required result,

$$\frac{d}{dx}\int_a^b f(x \pm u)\, g(u)\, du = \int_a^b f'(x \pm u)\, g(u)\, du\,;$$

(ii) *when $f(t)$ and $g(t)$ are both integrals for all the values of their arguments involved in the integral.*

In this case writing

$$\mathrm{F}(t) = \int_0^t f(t)\, dt,$$

we have, integrating by parts,

$$\int_a^b f(x+u)\, g(u) du = \Big[\mathrm{F}(x+u)\, g(u)\Big]_a^b - \int_a^b \mathrm{F}(x+u)\, g'(u)\, du,$$

which may be differentiated by case (i), giving,

$$\frac{d}{dx}\int_a^b f(x+u)\, g(u)\, du = \Big[g(u) f(x+u)\Big]_a^b - \int_a^b g'(u) f(x+u)\, du$$

$$= \int_a^b g(u) f'(x+u)\, du,$$

which is the required result. The same reasoning gives the required result when we have $f(x-u)$ instead of $f(x+u)$.

We have here supposed the limits of integration independent of x; in the contrary case we must, of course, add on corresponding terms to the result of integration under the integral sign.

§ 4. It will also be convenient to prefix the following two Lemmas:—

Lemma 1.—*If in a closed interval $(-\epsilon, \epsilon)$, $\mathrm{G}(x)$ is an integral which vanishes when $x = 0$, then the limits, as $n \to \infty$, of*

$$p \int_{-\epsilon}^{\epsilon} \mathrm{G}(x) \begin{matrix} \cos \\ \sin \end{matrix} px\, dx, \qquad (p = n + \tfrac{1}{2}),$$

can be made as small as we please by making ϵ conveniently small.

In fact, integrating by parts, the integral which appears vanishes by the Theorem of Riemann-Lebesgue, and the expression in square brackets is as small as we please.

LEMMA 2.—*If $tf(t)$ is an integral in the closed interval of periodicity and the coefficients of the derived series of the Fourier series of $f(t)$ converge to zero, then*

$$\operatorname*{Lt}_{n\to\infty}\left\{\int_{-\pi}^{\pi} pf(t)\cos pt\,dt - \sin p\pi\,[f(\pi)+f(-\pi)]\right\} = 0, \qquad (p = n+\tfrac{1}{2}). \qquad \text{(I)}$$

For, denoting the integral whose limit is considered by I,

$$\mathrm{I} = \int_{-\pi}^{\pi} pf(t)\cos nt\,dt - \int_{-\pi}^{\pi} pf(t)(1-\cos\tfrac{1}{2}t)\cos nt\,dt - \int_{-\pi}^{\pi} pf(t)\sin\tfrac{1}{2}t\sin nt\,dt.$$

The first integral approaches zero, being $1+1/2n$ times one of the coefficients of the derived series.

The second and third integrals are of the form

$$p\int_{-\pi}^{\pi} g(t)\begin{matrix}\cos\\ \sin\end{matrix}\, nt\,dt,$$

where $g(t)$ is an integral, since, by hypothesis $tf(t)$ is an integral.

Integrating by parts, this becomes

$$(1+1/2n)\left[\left(g(t)\begin{matrix}\sin\\ -\cos\end{matrix}\, nt\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} g'(t)\begin{matrix}\sin\\ -\cos\end{matrix}\, nt\,dt\right)$$

in which the integral approaches zero by the Theorem of Riemann-Lebesgue.

The bracket expression is zero when it involves the sine, and when it involves the cosine is

$$-[g(\pi)+g(-\pi)]\cos n\pi = -[f(\pi)+f(-\pi)]\cos n\pi,$$

whence, since $\cos n\pi = \sin p\pi$, and this term appears in (I) with minus sign, the required result follows.

Similarly

$$\operatorname*{Lt}_{n\to\infty}\int_{-\pi}^{\pi} pf(t)\sin pt\,dt = 0, \qquad (p = n+\tfrac{1}{2}). \qquad \text{(II)}$$

§ 4. The following auxiliary theorem is an immediate consequence of the known theory of Fourier series :—

THEOREM I.—*If* (i) *the associated function $F(u)$ of a Fourier series is an integral in the closed interval $(x-e,\, x+e)$ of its interval of periodicity $(-\pi, \pi)$, so that it has accordingly an absolutely integrable differential coefficient, $f(u)$, say, existing almost everywhere in the closed interval $(x-e, x+e)$; and if*

$$\text{(ii)}\ \mathrm{Q} = \frac{d}{dx}\int_{e}^{\pi}[\mathrm{F}(x+u)+\mathrm{F}(x-u)]\sin(n+\tfrac{1}{2})u\operatorname{cosec}\tfrac{1}{2}u\,du \to 0, \quad (n\to\infty),$$

then the upper and lower functions of the derived series of the Fourier series of F (u) *at the point* x *depend only on the properties of* $f(u)$ *in a neighbourhood of the point* x, *as small as we please, and will be the same as those of the Fourier series of any absolutely integrable function, which agrees with* $f(u)$ *in the closed interval* $(x-e, x+e)$.

Denote by s_n the nth partial summation of the derived series in question, then, by the usual theory,

$$s_n = \frac{1}{2\pi}\frac{d}{dx}\int_0^{\pi}[(\mathrm{F}\,x+u)+\mathrm{F}(x-u)]\sin(n+\tfrac{1}{2})u\,\mathrm{cosec}\,\tfrac{1}{2}u\,du.$$

and therefore, by condition (ii), the upper and lower limits of s_n are the same as those of

$$\frac{1}{2\pi}\frac{d}{dx}\int_0^{e}[\mathrm{F}(x+u)+\mathrm{F}(x-u)]\sin(n+\tfrac{1}{2})u\,\mathrm{cosec}\,\tfrac{1}{2}u\,du.$$

They are, therefore, by § 2, the same as the limits of

$$\frac{1}{2\pi}\int_0^{e}[f(x+u)+f(x-u)]\sin(n+\tfrac{1}{2})u\,\mathrm{cosec}\,\tfrac{1}{2}u\,du. \tag{1}$$

This proves the theorem. In fact, if $f_1(u)$ be any absolutely integrable function, agreeing with $f(u)$ in $(x-e, x+e)$, then, by the known theory,

$$\int_e^{\pi}[f_1(x+u)+f_1(x-u)]\sin(n+\tfrac{1}{2})u\,\mathrm{cosec}\,\tfrac{1}{2}u\,du \to 0, \qquad (n\to\infty);$$

and therefore the limits of the partial summations of the Fourier series of $f_1(u)$ at the point x are the same as those of the expression (1).

Cor. 1.—*If in condition* (ii) *the convergence to zero at a point or in an interval is uniform, the restricted Fourier series of* $f(x)$ *and the Fourier series of* $f_1(x)$ *will have the same peculiarities with regard to uniformity, or non-uniformity, of convergence, or oscillation, at the point, or in the interval.*

Cor. 2.—*Under the same circumstances as in Cor.* 1, *we may, in integration theorems involving integration term-by-term of a Fourier series when multiplied by a function* $g(x)$ *which is absolutely integrable, substitute for the Fourier series the restricted Fourier series of a function* $f(x)$ *which agrees with the associated function* $f_1(x)$ *of the Fourier series in a sub-interval, provided the range of integration be restricted to that sub-interval.*

In fact the difference between the nth partial summation of the restricted Fourier series and that of the Fourier series converges uniformly to zero, and therefore continues to do so when multiplied by any absolutely integrable function $g(x)$, and integrated term-by-term.

§ 5. We have now to transform condition (ii) of the theorem just given in a manner suitable for the purposes in hand.

THEOREM II.—(i) *If, outside the interval* $(x-e,\ x+e)$, $F(u)$ *is everywhere an integral, except at the finite number of points* $k_1, k_2\ldots, k_s$; *and if*

(ii) *for each* $r = 1, 2,\ldots, s$, *the expression*

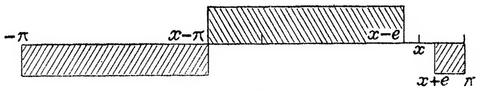
$$R = \int_{-\epsilon}^{\epsilon} F_r(u) \cos p\,(x_r-u)\,\tfrac{1}{2}p \operatorname{cosec} \tfrac{1}{2}(x_r-u)\,du$$

$$+F_r(\epsilon) \sin p\,(x_r-\epsilon) \operatorname{cosec} \tfrac{1}{2}(x_r-\epsilon) - F_r(-\epsilon) \sin p\,(x_r+\epsilon) \operatorname{cosec} \tfrac{1}{2}(x_r+\epsilon),$$

where $F_r(u)$ *denotes* $F(u+k_r)$, *and* $x_r = x-k_r$ *and where* p *has, for brevity, been written for* $n+\frac{1}{2}$, *has, as* $n\to\infty$, *limits which are as small as we please,* ϵ *being conveniently small, then the expression*

$$Q = \frac{d}{dx}\int_e^{\pi}[F(x+u)+F(x-u)] \sin(n+\tfrac{1}{2})\,u \operatorname{cosec} \tfrac{1}{2}u\,du \to 0, \qquad (n\to\infty).$$

For simplicity we may suppose that the extremities of the interval of periodicity are not included among the points k_r. No loss of generality is hereby occasioned.



For definiteness, suppose x positive.

The points k_r fall into two classes, those which lie in $(x-\pi,\ x-e)$, and those which lie in $(x+e,\ \pi)$ or in $(-\pi, x-\pi)$ (see figure).

As u increases from e to π, $x-u$ will pass over points of the former class only, and $(x+u)$ over points of the latter class only.

First let k_r' denote one of the first class, and consider the expression

$$Q_1 = \frac{d}{dx}\int_e^{\pi} F(x-u) \sin(n+\tfrac{1}{2})\,u \operatorname{cosec} \tfrac{1}{2}u\,du.$$

Then, supposing there are s' such points, we divide up the interval $(e,\ \pi)$ into $2s'+1$ intervals. These intervals consist of (i) the two end-intervals $(e,\ x-k_1'-\epsilon)$ and $(x-k_{s'}'+\epsilon,\ \pi)$, and other corresponding intervals which do not contain the points $x-k_r'$, (ii) intervals of the type $(x-k_r'-\epsilon,\ x-k_r'+\epsilon)$. Consider the portion of Q_1 due to an interval of the first class. This is of the type

$$\frac{d}{dx}\int_{x-\alpha+\epsilon}^{x-\beta-\epsilon} F(x-u) \sin(n+\tfrac{1}{2})\,u \operatorname{cosec} \tfrac{1}{2}u\,du$$

the integrand having no singularities with respect to $x-u$ or u, so that, by § 2, this portion of Q_1 may be differentiated in the usual way, giving us three terms, one of which is the integral obtained by omitting d/dx in

the preceding expression and writing f for F, and accordingly has the unique limit zero, and the other two are :

$$F(\beta+\epsilon)\sin p(x-\beta-\epsilon)\operatorname{cosec}\tfrac{1}{2}(x-\beta-\epsilon)$$
$$-F(\alpha-\epsilon)\sin p(x-\alpha+\epsilon)\operatorname{cosec}\tfrac{1}{2}(x-\alpha+\epsilon).$$

In the case of the two end-intervals which belong to this class, one only of these two latter terms, of course, appears.

Next consider the portion of Q_1 due to an interval of the second class. This is of the type

$$\frac{d}{dx}\int_{x-\alpha-\epsilon}^{x-\alpha+\epsilon} F(x-u)\sin pu\operatorname{cosec}\tfrac{1}{2}u\,du,$$

which may be written

$$\frac{d}{dx}\int_{-\epsilon}^{\epsilon} F(t+\alpha)\sin p(x-\alpha-t)\operatorname{cosec}\tfrac{1}{2}(x-\alpha-t)\,dt,$$

where, by § 2, the differentiation may be performed under the integral sign. We thus get two terms, one of which has the unique limit zero, by the Theorem of Riemann-Lebesgue, while the other is

$$p\int_{-\epsilon}^{\epsilon} F(t+\alpha)\cos p(x-\alpha-t)\operatorname{cosec}\tfrac{1}{2}(x-\alpha-t)\,dt.$$

Collecting our results so far, and bearing in mind that, in the statement of condition (ii), x_r stands for $(x-k_r)$, and $F_r(u)$ for $F(u+k_r)$, we see that, as far as Q_1 is concerned, the singular points of the first of the two classes, into which we had divided them, give us terms precisely of the type stated in the enunciation of the theorem.

Next consider

$$Q_2 = \frac{d}{dx}\int_{\epsilon}^{\pi} F(x+u)\sin pu\operatorname{cosec}\tfrac{1}{2}u\,du.$$

We shall have work of precisely the same kind to perform as before in dealing with the first class of intervals. In the second class we shall write,

$$u = k_r''-x+t,$$

and paying attention to the sign of ϵ, we shall find that, for Q_2 also, the condition (ii) contains expressions to which we are actually led by our process. Thus our theorem is proved.

§ 6. It should be noticed that in the preceding theorem we have only used the condition that the singularities of $F(u)$ are point singularities, in demanding merely that the expressions R can be made as small as we please, by taking ϵ conveniently small. If we require R to converge to zero, the singularities may be any we please inside the corresponding intervals.

In the next auxiliary theorem, about to be given, it is, however, essential that the points at which $F(x)$ is not an integral should be finite in number.

THEOREM III.—(i) *If outside the interval* $(x-e, x+e)$, $F(u)$ *is everywhere an integral, except at the finite number of points* $k_1, k_2, \ldots, k_s$;

(ii) $tF(t+k_r)$ *is for each value of* $r = 1, 2, \ldots, s$, *and an appropriate sufficiently small range of values of* t, *including* $t = 0$, *an integral which converges to zero, as* $t \to 0$;

(iii) *The expressions*

$$p\int_{-\epsilon}^{\epsilon} F(t+k_r) \begin{matrix}\cos\\ \sin\end{matrix} pt\,dt \mp F(\epsilon+k_r) \begin{matrix}\sin\\ \cos\end{matrix} p\epsilon - F(-\epsilon+k_r) \begin{matrix}\sin\\ \cos\end{matrix} p\epsilon$$

have, as $n \to \infty$, *limits all of which may be made as small as we please, by making* ϵ *conveniently small; then the expressions which occur in Theorem* II (ii), *viz.*,

$$R = \int_{-\epsilon}^{\epsilon} F_r(u) \cos p(x_r-u)\, \tfrac{1}{2}p \operatorname{cosec} \tfrac{1}{2}(x_r-u)$$

$$+F(\epsilon) \sin p(x_r-\epsilon) \operatorname{cosec} \tfrac{1}{2}(x_r-\epsilon) - F(-\epsilon) \sin p(x_r+\epsilon) \operatorname{cosec} \tfrac{1}{2}(x_r+\epsilon)$$

have limits which are as small as we please, ϵ *being conveniently small.*

We have to consider the limiting value, or values, of expressions of the type

$$p\int_{-\epsilon}^{\epsilon} F_r(u) \cos p(x_r-u) \operatorname{cosec} \tfrac{1}{2}(x_r-u)\, du + F_r(\epsilon) \sin p(x_r-\epsilon) \operatorname{cosec} \tfrac{1}{2}(x_r-\epsilon)$$

$$-F_r(-\epsilon) \sin p(x_r+\epsilon) \operatorname{cosec} \tfrac{1}{2}(x_r+\epsilon) = A+B+C, \text{ say.}$$

Write $\qquad \operatorname{cosec} \tfrac{1}{2}(x_r-u) = \operatorname{cosec} \tfrac{1}{2}x_r + u\, g(x_r, u),$

then

$$B+C = \{F_r(\epsilon) \sin p(x-\epsilon) - F_r(-\epsilon) \sin p(x_r+\epsilon)\} \operatorname{cosec} \tfrac{1}{2}x_r$$

$$+\{\epsilon F(\epsilon)\, g(x_r, \epsilon) \sin p(x_r-\epsilon) + \epsilon F_r(-\epsilon)\, g(x_r, -\epsilon) \sin p(x_r+\epsilon)\},$$

where the quantity inside the second curly bracket is as small as we please, when ϵ is chosen conveniently small, since, by hypothesis (ii), $uF_r(u)$ approaches zero, when $u \to 0$, and $g(x, \pm\epsilon) \sin p(x \mp \epsilon)$ is bounded.

Thus we may replace $B+C$ by

$$\sin px \cos p\epsilon_r \operatorname{cosec} \tfrac{1}{2}x_r [F_r(\epsilon) - F_r(-\epsilon)]$$

$$-\cos px_r \sin p\epsilon \operatorname{cosec} \tfrac{1}{2}x_r [F_r(\epsilon) + F_r(-\epsilon)]. \quad (1)$$

Let us write

$$A = L \cos px_r + M \sin px_r,$$

where

$$L = p\int_{-\epsilon}^{\epsilon} F_r(t) \cos pt \operatorname{cosec} \tfrac{1}{2}(x_r-t)\, dt = \operatorname{cosec} \tfrac{1}{2}x_r \int_{-\epsilon}^{\epsilon} p\, F_r(t) \cos pt\, dt$$

$$+\int_{-\epsilon}^{\epsilon} pt\, F_r(t)\, g(x_r, t) \cos pt\, dt.$$

The integral last written down being as small as we please, by Lemma 1, since $t\mathrm{F}_r(t)$ and $g(x_r, t)$ are both integrals, so that $t\mathrm{F}_r(t)\, g(x_r, t)$ is an integral, and has, moreover, like $t\mathrm{F}_r(t)$, the unique limit zero, when $t \to 0$.

Combining this with (1), we see that $\mathrm{A}+\mathrm{B}+\mathrm{C}$ has limits which differ by as little as we please from those of an expression of the form

$$(\mathrm{L}' \cos px_r + \mathrm{M}' \sin px_r) \operatorname{cosec} \tfrac{1}{2} x_r$$

where

$$\mathrm{L}' = p \int_{-\epsilon}^{\epsilon} \mathrm{F}_r(t) \cos pt\, dt - \sin p\epsilon\, [\mathrm{F}_r(\epsilon) + \mathrm{F}_r(-\epsilon)],$$

and, transforming M in the same way as we transformed L above,

$$\mathrm{M}' = p \int_{-\epsilon}^{\epsilon} \mathrm{F}_r(t) \sin pt\, dt + \cos p\, [\mathrm{F}_r(\epsilon) - \mathrm{F}_r(-\epsilon)].$$

But, by the hypothesis (ii), the expressions L' and M' have limits as small as we please, for all values of $r = 1, 2, \ldots, s$. Thus each of the expressions R with which we started has limits as small as we please, which proves the theorem.

Cor.—*In condition* (iii) p *may be replaced by* n, *without disturbing the validity of the theorem.*

As regards the second and third members of the expression, this is almost obvious; in fact, taking, for instance, the sine,

$$\mathrm{F}(t+k_r) \sin pt$$

differs from

$$\mathrm{F}(t+k_r) \sin nt$$

by

$$\mathrm{F}(t+k_r) \sin nt\, (1-\cos \tfrac{1}{2}t) + \mathrm{F}(t+k_r) \sin \tfrac{1}{2}t \cos nt,$$

which, by the last statement of condition (iii), converges to zero, as $t \to 0$, and therefore has, when $t = \epsilon$, a value as small as we please.

As regards the integral constituting the first term of the expression, we may clearly replace the p outside the integral by n, since, by the Theorem of Riemann-Lebesgue, we shall only alter the term by a quantity which converges to zero, as n or p becomes infinite. Suppose this change already made. If we now change the p inside the integral into n, we have a difference in value given by the sum of two expressions of the form

$$n \int_{-\epsilon}^{\epsilon} t\, \mathrm{F}(t+k_r)\, h(t) \begin{array}{c} \cos \\ \cdot \\ \sin \end{array} nt\, dt$$

where $h(t)$ is an integral. These, by Lemma 1 of § 4, have, for a sufficiently small ϵ, limits as small as we please. For $t\mathrm{F}(t+k_r)\, h(t)$ is an integral which approaches zero, as $t \to 0$, and, therefore, as shown in that lemma, integrating by parts, and, using the Theorem of Riemann-Lebesgue, our result follows.

§ 7. We now come to our fundamental auxiliary theorem.

THEOREM IV.—*The (ordinary) upper and lower functions of the first derived series of the Fourier series of* F (x) *at a particular point* x *(other than the points* $k_1, k_2, \ldots, k_s$, *to be immediately specified) depend only on the nature of* F (t) *in a neighbourhood enclosing the point* $t = x$ *considered, as small as we please, provided the following conditions be satisfied:—*

(i) *Except in an interval which contains one at least of a certain finite number of points,* $k_1, k_2, \ldots, k_s$, F (t) *is an absolutely convergent integral.*

(ii) $t\mathrm{F}(k_r+t)$ *is an absolutely convergent integral for* $r = 1, 2, \ldots, s$, *in a certain interval containing* $t = 0$, *and converges to zero, when* $t \to 0$.

(iii) *If* $q_r(t)$ *denote any function which, except in a certain sub-interval of the interval* $(-\pi, \pi)$, *surrounding* $t = k_r$, *is a periodic integral, and in that exceptional sub-interval is equal to* F (t), *then the coefficients of the derived series of the Fourier series of* $q_r(t)$ *converge to zero, for* $r = 1, 2, \ldots, s$.

For, if the conditions (iii) hold,

$$\operatorname*{Lt}_{n\to\infty} \frac{n}{\pi}\int_{-\pi}^{\pi} q_r(t) \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix}\, nt\, dt = 0.$$

Hence
$$\operatorname*{Lt}_{n\to\infty} n\int_{-\pi}^{\pi} q_r(t) \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix}\, n(t-k_r)\, dt = 0.$$

Therefore, since $q_r(t)$ is periodic,

$$\operatorname*{Lt}_{n\to\infty} n\int_{-\pi}^{\pi} q_r(t+k_r) \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix}\, nt\, dt = 0 \tag{1}$$

Now

$$n\int_{\epsilon}^{\pi} q_r(t+k_r) \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix}\, nt\, dt = \Big[q_r(t+k_r) \begin{matrix}\sin\\ -\cos\end{matrix}\, nt\Big]_{\epsilon}^{\pi} - \int_{\epsilon}^{\pi} q_r(t+k_r) \begin{matrix}\sin\\ -\cos\end{matrix}\, nt\, dt,$$

where the integral on the right vanishes when $n \to \infty$, by the Theorem of Riemann-Lebesgue. Hence, as $n \to \infty$,

$$n\int_{\epsilon}^{\pi} q_r(t+k_r) \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix}\, nt\, dt - q_r(\pi+k_r) \begin{matrix}\sin\\ -\cos\end{matrix}\, n\pi + q_r(\epsilon+k_r) \begin{matrix}\sin\\ -\cos\end{matrix}\, n\epsilon \to 0. \tag{2}$$

In like manner

$$n\int_{-\pi}^{-\epsilon} q_r(t+k_r) \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix}\, nt\, dt - q_r(-\epsilon+k_r) \begin{matrix}-\sin\\ \cos\end{matrix}\, n\epsilon + q_r(-\pi+k_r) \begin{matrix}-\sin\\ \cos\end{matrix}\, n\pi \to 0. \tag{3}$$

Also
$$n\int_{-\epsilon}^{\epsilon} q_r(t+k_r) \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix}\, nt\, dt = n\int_{-\epsilon}^{\epsilon} \mathrm{F}(t+k_r) \begin{matrix}\sin\\ \cos\end{matrix}\, nt. \tag{4}$$

Adding (2), (3) and (4), and comparing with (1), we get, finally,

$$\left\{n\int_{-\epsilon}^{\epsilon} \mathrm{F}(t+k_r) \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix}\, nt - [q\,\mathrm{F}_r(\epsilon+k_r) \pm q\,\mathrm{F}_r(-\epsilon+k_r)] \begin{matrix}\cos\\ \cdot\\ \sin\end{matrix}\, n\epsilon\right\} \to 0, \tag{5}$$

since $\qquad \sin n\pi = 0, \qquad$ and $\qquad q_r(\pi+k_r) = q_r(-\pi+k_r),$

and $\qquad q_r(\epsilon+k_r) = \mathrm{F}(\epsilon+k_r), \qquad q_r(-\epsilon+k_r) = \mathrm{F}(-\epsilon+k_r).$

Now, as in the preceding corollary, we may in (5) replace n by p, where $p = n+\frac{1}{2}$. The conditions (i), (ii), and (iii) of Theorem III of § 6 are therefore satisfied. Hence, by Theorems II and III,

$$\operatorname*{Lt}_{n\to\infty} \mathrm{I}_n = \operatorname*{Lt}_{n\to\infty} \frac{d}{dx}\int_c^{\pi} [\mathrm{F}(x+u)+\mathrm{F}(x-u)] \sin(n+\tfrac{1}{2})u \operatorname{cosec}\tfrac{1}{2}u\, du = 0.$$

This, by Theorem I, proves the theorem.

§ 8. From Theorem IV of the preceding article, we conclude easily that whole classes of restricted Fourier series of the kind contemplated actually exist. In a recent communication to the Society, I have virtually obtained sufficient conditions for the fulfilment of condition (iii) of Theorem IV above. We proceed first to show that these sufficient conditions ensure at the same time the fulfilment of condition (ii).

Lemma 3.—*If* F (u) *is an integral in every interval not containing the origin, and if any one of the following conditions holds in some interval containing the origin,*

(i) $\qquad u\mathrm{F}'(u)$ *is bounded;*

(ii) $\qquad \mathrm{F}(u) - \dfrac{1}{u}\displaystyle\int_0^u \mathrm{F}(u)\,du$ *is an integral;*

(iii) $\qquad u^2\mathrm{F}''(u)$ *is bounded;*

then $u\mathrm{F}(u)$ *is an integral in the whole interval of periodicity, and approaches zero when* $u \to 0$.

We have, in fact, if G (u) is the integral of F (u),

$$\frac{d}{du}[u\mathrm{F}(u)] = \mathrm{F}(u)+u\mathrm{F}'(u) = \frac{d}{du}\mathrm{G}(u)+u\mathrm{F}'(u),$$

and, therefore, in case (i), the differential coefficient of $u\mathrm{F}(u)-\mathrm{G}(u)$ is bounded, so that $u\mathrm{F}(u)-\mathrm{G}(u)$ is an integral, and, therefore, $u\mathrm{F}(u)$ is an integral, which, by the theory of indeterminate forms, has the same limit as $-u^2\mathrm{F}'(u)$, that is zero.

In case (ii), if we multiply by u, we still get an integral, which accordingly approaches zero, when $u \to 0$, hence the required result follows.

In case (iii), we remark that, by the theory of indeterminate forms,

$$\mathrm{F}'(u)/(1/u)$$

has its limits among those of

$$\mathrm{F}''(u)/(-1/u^2),$$

so that (iii) becomes a special case of (i).

§ 9. We now give the conditions alluded to in the previous article, which

ensure simultaneously the holding of the conditions (ii) and (iii) of Theorem IV. Taken in conjunction with that theorem, they give us the main results of the paper, stated in the Introduction.

THEOREM A.—*The conditions* (ii) *and* (iii) *of Theorem* IV *are satisfied, for a particular value of* r *if, for that value of* r, *we have, as* $u \to 0$,

(*a*) $$F(k_r+u)-F(k_r-u) \to 0; \text{ and}$$

(*b*) $$u\frac{d}{du}[F(k_r+u)+F(k_r-u)] \to 0;$$

and if further

(*c*) $uF'(k_r+u) = uf(k_r+u)$ *is for some interval surrounding* $u = 0$, *a function of bounded variation;*

or, more generally, if we have, in addition to (*a*), *the following:—*

(*b'*) $$\frac{1}{u}\int_0^u t[f(t+k_r)-f(k_r-t)]\,dt \to 0, \text{ when } u \to 0;$$

(*c'*) $F(u+k_r)-\frac{1}{u}\int_0^u F(t+k_r)\,dt$ *is an integral for an interval containing* $u = 0$.

Condition (*a*) secures that the odd function $F(k_r+u)-F(k_r-u)$ has zero as limit, when $u \to 0$, and (*c*) secures that u times the differential coefficient of the same odd function is a function of bounded variation. Hence we may apply Theorem 4 of my paper cited above, and deduce the convergence to zero of the coefficients of the derived series of the extended odd function, which is elsewhere an integral, and in the given small interval agrees with the above odd function.

Similarly conditions (*b*) and (*c*) secure that u times the differential coefficient of the even function $F(k_r+u)+F(k_r-u)$ is a function of bounded variation, which converges to zero, as $u \to 0$. Thus by Theorem 3 of the same paper, the coefficients of the derived series of the extended even function converge to zero.

Hence the coefficients both of the cosine and the sine terms of the derived series of the Fourier series of a function which is equal to $F(u)$ in the given small interval containing $u = k_r$, and is elsewhere an integral, converge to zero. Hence condition (iii) of Theorem IV is satisfied.

With regard to condition (ii), this is, by Lemma 3, satisfied, as by condition (*c*) of the present theorem $uF'(k_r+u)$ is bounded.

This proves the theorem when (*a*), (*b*), and (*c*) are satisfied.

To see that (*b'*) and (*c'*) may take the place of (*b*) and (*c*), we remark first that, by Lemma 3, $uF(u+k_r) = uF_r(u)$ approaches zero, as $u \to 0$, and that, therefore, from the equation

$$\int_\epsilon^u uf_r(u)\,du = uF_r(u) - \int_\epsilon^u F_r(u)\,du - F_r(\epsilon),$$

we may deduce that

$$\int_0^u uf_r(u)\,du = u\mathrm{F}_r(u) - \int_0^u \mathrm{F}_r(u)\,du,$$

and, therefore, that the left-hand side of the last equation, being the difference of two integrals, is an ordinary Lebesgue integral.

Moreover we see that

$$\frac{1}{u}\int_0^u uf(u)\,du$$

is itself an ordinary Lebesgue integral.

The rest of the argument is precisely analogous to that employed in dealing with the set of conditions (a), (b), and (c) except that, instead of using Theorems 3 and 4 of the paper cited as basis, we now use the more general Theorems 5 and 6 of that same paper.

§ 10. The following theorem is based on Theorems 7 and 8 of the paper previously cited, and in its most general form, on a slight generalisation of the result of my 'Comptes Rendus' paper, there utilised.

THEOREM B.—*Retaining conditions* (a) *of Theorem* A, *we may replace* (b) *and* (c) *by the following* :—

(b'') $u^2\mathrm{F}''(u+k_r)$ *is bounded in some interval surrounding* $u = 0$, *and* $u\mathrm{F}'(u+k_r)$ *has a unique limit as* $u \to 0$, F'' *denoting a derivate of* F'.

In fact, by Lemma 3, condition (ii) of Theorem IV is satisfied.

More generally we may replace (b) *and* (c) *by the following* :—

(b''') $\dfrac{1}{u}\displaystyle\int \left| d\,(t^2\mathrm{F}'(t+k_r)\right|$ *is bounded in some interval surrounding* $u = 0$, *and* $u\mathrm{F}'(u+k_r)$ *has a unique limit, as* $u \to 0$.

In fact, by Lemma 3, condition (ii) of Theorem IV is satisfied. Also considering (c''), if we write

$$g(u) = \frac{d}{du}\left[u^2\mathrm{F}'(u+k_r)\right] = 2uf(k_r) + u^2 f'(u+k_r),$$

we see that $g(u)$ is bounded, say numerically less than B.

Further
$$uf(u+k_r) = \frac{1}{u}\int_0^u g(t)\,dt,$$

where
$$\frac{1}{u}\int_0^u g(t)\,dt \leq \mathrm{B},$$

and is accordingly bounded.

We can accordingly employ Theorems 7 and 8 of the paper already cited, and the required result follows.

To prove the remaining part of the theorem, we have merely to use the condition that

$$\frac{1}{u}\int_0^u \left| du\,[\phi(x+u)+\phi(x-u)]\right|$$

should be bounded, instead of the slightly less general condition that

$$\frac{1}{u}\int_0^u \frac{d}{du}\,u\,[\phi(x+u)+\phi(x-u)]\,du$$

in the test for the convergence of the Fourier series of a function $\phi(u)$ and of its allied series, quoted in the paper cited from the 'Comptes Rendus.'*

§ 11. In the Theorems II–IV we have had in view convergence at a point only. That the convergence is uniform, or, more generally, that the quantities have limits which are bounded functions of x, is immediate. Thus we have the following theorem :—

THEOREM V.—*The expression* $\dfrac{d}{dx}\displaystyle\int_e^\pi [F(x+u)+F(x-u)]\sin pu \operatorname{cosec}\tfrac{1}{2}u\,du$ *of Theorem* I *converges uniformly to zero in any closed interval internal to a completely open interval, in which the conditions of Theorem* IV *hold.*

We have, in fact, only to examine the proofs of Theorems II, III, and IV to see firstly that, for such a closed interval, there is a finite upper and a finite lower bound to each of the factors which involve x, *e.g.*,

$$\cos p\,(x-\alpha-t)\operatorname{cosec}\tfrac{1}{2}(x-\alpha-t),$$

or $g(x, \epsilon)$ in Theorem III; and, secondly, that wherever the Theorem of Riemann-Lebesgue is used, the convergence is, by a known property of the integral in question, uniform convergence to zero for values of x in our closed interval. Finally, where the argument introduces a small quantity ϵ, which, when chosen conveniently small, ensures certain auxiliary quantities being as small as we please, we see that these auxiliary quantities may be made less than an assigned small quantity independent of x, since, x being a point of a closed interval inside a completely open interval in which the conditions hold, ϵ may be chosen independent of x, and, by the point first referred to, the auxiliary quantity in question may be made to depend only on ϵ. Thus the theorem is proved.

§ 12. Referring then to Cor. 2 of Theorem I, we at once obtain the following results in the theory of term-by-term integration of restricted Fourier series when multiplied by another function :—

THEOREM C.—*If $f(x)$ is a function which in a certain sub-interval of the*

* The proof of the extended form of test is almost exactly the same as that given in the 'Comptes Rendus.' It is given *in extenso* in a paper on "The Convergence of the Derived Series of a Fourier Series," cited above.

interval of periodicity has its square summable, and $g(x)$ *is another function whose square is summable, then, if we multiply the restricted Fourier series of* $f(x)$ *by* $g(x)$, *we may integrate term by term over the sub-interval, and the result will be the integral of the product* $f(x)\, g(x)$ *over the same interval, provided only that—*

(i) *the restricted nature of the series is due to a finite number of points at which the primitive function* $F(x)$ *of which* $f(x)$ *is a derivate is not an integral ;*

(ii) *at such points* k_r, $(r = 1, 2, \ldots, s)$, $(x-k_r)\, F(x)$ *is an integral ;*

(iii) *if* q_r *denote any function which, except in a certain sub-interval of the interval* $(-\pi, \pi)$ *surrounding* $x = k_r$, *is a periodic integral, and in that exceptional sub-interval is equal to* $F(x)$, *then the coefficients of the derived series of the Fourier series of* $q_r(x)$ *converge to zero for* $r = 1, 2, \ldots, s$.

THEOREM D.—*If* $f(x)$ *is a function which, in a certain sub-interval of the interval of periodicity has bounded variation, and* $g(x)$ *is summable, then, if we multiply the restricted Fourier series of* $f(x)$ *by* $g(x)$, *and integrate term by term over the sub-interval, the result will be the integral of the product* $f(x)\, g(x)$ *over the same sub-interval, provided only the same conditions hold as in Theorem C.*

THEOREM E.—*If* $g(x)$ *is any function of bounded variation, then, if we multiply the restricted Fourier series of any function* $f(x)$ *by* $g(x)$, *we may integrate term by term over the sub-interval* to which it is restricted, and the result will be the integral of the product* $f(x)\, g(x)$ *over the same sub-interval, provided only the same conditions hold as in Theorem C.*

In these theorems, as elsewhere, the integration is supposed to be over a closed interval. By the term therefore "integration over the sub-interval," which is itself necessarily open, we mean that we may integrate between any two points of that open interval.

We may, of course, substitute for the conditions (i), (ii), (iii) any of those sets of conditions stated in Theorems A and B as sufficient to ensure their fulfilment.

It should be noticed that in Theorem D we use the fact that the convergence of the restricted Fourier series in an interval to which it is restricted is necessarily bounded, being of the same character as that for a proper Fourier series, in virtue of Cor. 2 to Theorem I.

* In the present communication, as in the paper which preceded it, I have confined my attention to first derived series of Fourier series. It is open to us also to consider the circumstances under which the coefficients of the second or higher derived series converge to zero, while the series themselves converge at isolated points, or throughout intervals. No new principles are introduced into the consideration of such higher derived series ; I have therefore, thought it undesirable to extend the length of my communications by dealing with them.